

APPROXIMATION PROPERTIES FOR NONCOMMUTATIVE L^p -SPACES OF HIGH RANK LATTICES AND NONEMBEDDABILITY OF EXPANDERS

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ABSTRACT. This article contains two rigidity type results for $\mathrm{SL}(n, \mathbb{Z})$ for large n that share the same proof. Firstly, we prove that for every $p \in [1, \infty]$ different from 2, the noncommutative L^p -space associated with $\mathrm{SL}(n, \mathbb{Z})$ does not have the completely bounded approximation property for sufficiently large n depending on p .

The second result concerns the coarse embeddability of expander families constructed from $\mathrm{SL}(n, \mathbb{Z})$. Let X be a Banach space and suppose that there exist $\beta < \frac{1}{2}$ and $C > 0$ such that the Banach-Mazur distance to a Hilbert space of all k -dimensional subspaces of X is bounded above by Ck^β . Then the expander family constructed from $\mathrm{SL}(n, \mathbb{Z})$ does not coarsely embed into X for sufficiently large n depending on X .

More generally, we prove that both results hold for lattices in connected simple real Lie groups with sufficiently high real rank.

1. INTRODUCTION

The aim of this article is to prove two rigidity type results for $\mathrm{SL}(n, \mathbb{Z})$ for large n , and more generally for high rank lattices, that share the same proof.

Let Γ be a countable discrete group, and let $L(\Gamma)$ be the group von Neumann algebra of Γ . For every $p \in (1, \infty)$, we can form the noncommutative L^p -space associated with Γ by taking the completion of $L(\Gamma)$ with respect to the norm $\|\cdot\|_p$ coming from the trace τ on $L(\Gamma)$ via $\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$. A noncommutative L^p -space has the completely bounded approximation property (CBAP) if there exists a net of finite-rank maps on it that is uniformly bounded in the completely bounded norm and that approximates the identity map pointwise. We refer to Section 2 for more details on group von Neumann algebras, noncommutative L^p -spaces and the CBAP.

Until recently, no explicit examples of noncommutative L^p -spaces without the CBAP were known. The first explicit examples of such spaces were given by Laforgue and the second named author in [20]. They proved that for $n \geq 3$ and $p \in [1, \frac{4}{3}) \cup (4, \infty]$, the noncommutative L^p -spaces $L^p(L(\mathrm{SL}(n, \mathbb{Z})))$ do not have the CBAP. This result was extended in [16] and [12]. From these two articles, it follows that for every lattice Γ in a connected simple real Lie group with real rank at least 2 and every $p \in [1, \frac{12}{11}) \cup (12, \infty]$, the space $L^p(L(\Gamma))$ does not have the CBAP. In fact, this is even true for $p \in [1, \frac{10}{9}) \cup (10, \infty]$, as follows from [17, Appendix A].

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The first main result of this article deals with the CBAP for the noncommutative L^p -spaces associated with the group $\mathrm{SL}(n, \mathbb{Z})$.

Theorem 1.1. Let $n \geq 3$, and let $r \geq 2n - 3$. Then the noncommutative L^p -space $L^p(L(\mathrm{SL}(r, \mathbb{Z})))$ does not have the CBAP for $p \in [1, 2 - \frac{2}{n}) \cup (2 + \frac{2}{n-2}, \infty]$.

Note that this result extends the result of Lafforgue and the second named author mentioned above. An analogue of Theorem 1.1 in the non-Archimedean setting was already known from [20]. Theorem 1.1 was therefore expected. In fact, it answers one of the questions left open in [20]. The following essential question remains open.

Question: Does $L^p(L(\mathrm{SL}(3, \mathbb{Z})))$ have the CBAP for some $p \neq 2$?

The importance of this question is its relation with the (non-)isomorphism problem of the group von Neumann algebras of $\mathrm{PSL}(n, \mathbb{Z})$ for different values of $n \geq 3$, which is a deep open problem going back to [5]. Indeed, an affirmative answer to the question above would imply that $L(\mathrm{SL}(3, \mathbb{Z}))$ is not isomorphic to $L(\mathrm{SL}(n, \mathbb{Z}))$ for certain values of $n \geq 4$.

Remark 1.2. From Theorem 1.1, it follows that for every $p \neq 2$, the noncommutative L^p -space associated with any countable discrete group containing $\mathrm{SL}(n, \mathbb{Z})$ as a subgroup for every $n \geq 3$ does not have the CBAP. There are several ways to construct such a group. In particular, there are finitely presented examples [4].

The proof of Theorem 1.1 proceeds in the same way as the proof of [20, Theorem B], part of which was itself inspired by [18]. However, the computations needed in this article are significantly more involved. The idea is as follows. Firstly, we use a result proved in [20], asserting that if Γ is a lattice in a locally compact group G and $L^p(L(\Gamma))$ has the CBAP for some $p \in (1, \infty)$, then G has the so-called $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ for that value of p (see Section 2 for the definition of the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ and details). The $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ was introduced in [20] exactly to this purpose. The strategy then becomes to show the failure of the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ for G . The main new ingredient (Proposition 3.1) is a result on harmonic analysis on the sphere \mathbb{S}^{n-1} for $n \geq 3$, for which a careful study of the spherical functions for the Gelfand pair $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is needed.

A more general version of Theorem 1.1 for lattices in connected simple real Lie groups with high rank is obtained as well (see Theorem 4.7).

We now move to the second main result of this article. Let S be a symmetric finite generating set of $\mathrm{SL}(n, \mathbb{Z})$, and for $i \geq 1$, let $\pi_i: \mathrm{SL}(n, \mathbb{Z}) \rightarrow \mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$ denote the natural surjective homomorphism. As observed by Margulis, the Cayley graphs $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ form an expander family (see Section 2 for the definition of expander family). It is an open problem whether for (say) $n = 3$, this family embeds coarsely in any superreflexive Banach space (see [18], [19], [29], [22], [30] for related results). A Banach space X is superreflexive if every Banach space finitely representable in X is reflexive. Our contribution to this question is that, modulo a classical open problem in Banach space theory, a superreflexive Banach space does not coarsely contain $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ for n large enough. In fact, we prove a non-embeddability result for these expander families in any (not necessarily superreflexive) Banach space satisfying a certain geometric criterion, to be made

precise below. The notions from the geometry of Banach spaces that we use in what follows are recalled in Section 5.3.

For every Banach space X , consider the sequence of real numbers defined by

$$d_k(X) = \sup\{d(E, \ell_{\dim E}^2) \mid E \subset X, \dim E \leq k\},$$

where d denotes the Banach-Mazur distance. It is always true that $d_k(X) \leq k^{\frac{1}{2}}$, and if X has type > 1 (in particular, if X is superreflexive), then $d_k(X) = o(k^{\frac{1}{2}})$. Our results will apply to the Banach spaces X for which

$$(1) \quad \exists \beta < \frac{1}{2}, \exists C > 0 \text{ such that } d_k(X) \leq Ck^\beta \text{ for all } k \geq 1.$$

This includes the spaces of type 2 and, more generally, the ones of type p and cotype q satisfying $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$. It is a well-known open problem whether all Banach spaces of type > 1 satisfy (1) (see Section 5).

Our second result is as follows (see Theorem 5.11 for a more general statement for Schreier graphs coming from high rank lattices).

Theorem 1.3. Let X be a Banach space satisfying (1). Then the expander family $(\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z}), \pi_i(S))_{i \geq 1}$ does not coarsely embed into X for sufficiently large n depending on X .

First we prove that if X satisfies (1) and is superreflexive, then for n sufficiently large, the group $\mathrm{SL}(n, \mathbb{R})$ has a version of property (T) relative to X that was defined in [18, Section 4]. In order to prove this, we find a certain sequence of compactly supported measures m_k on $\mathrm{SL}(n, \mathbb{R})$ such that $(\pi(m_k))_k$ converges for every isometric representation π on X . The next step is to identify the limit of the sequence $(\pi(m_k))_k$ with a projection onto the $\pi(\mathrm{SL}(n, \mathbb{R}))$ -invariant vectors. Here we cannot use the methods of [18] (and hence we cannot prove Lafforgue's strong property (T) for $\mathrm{SL}(n, \mathbb{R})$ relative to X); instead we use a version of the Howe-Moore property for $\mathrm{SL}(n, \mathbb{R})$ as proved by Shalom (see [1, Theorem 9.1]). This is where the superreflexivity assumption is used. Afterwards, we explain why the superreflexivity condition is, in fact, not necessary to get Theorem 1.3, by showing that for a general Banach space X satisfying (1) and for n sufficiently large, the group $\mathrm{SL}(n, \mathbb{R})$ has a version of property (T) with respect to a certain class of representations on X -valued L^2 -spaces.

The version of property (T) discussed above passes from a locally compact group to its lattices [18], [19]. Also, if a group has this property with respect to a superreflexive Banach space X , then the group has property (T_X) as defined in [1]. The following result is immediate.

Theorem 1.4. Let X be a superreflexive Banach space satisfying (1). Then the groups $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{Z})$ have property (T_X) for sufficiently large n depending on X .

In fact, we prove that a “non-uniform” version of the above property is equivalent to property (T_X) (see Proposition 5.1).

This article is organized as follows. After recalling some preliminaries in Section 2, we obtain the aforementioned result on harmonic analysis on \mathbb{S}^{n-1} in Section 3. In Section 4, we use this result to prove Theorem 1.1. In Section 5, we show how the proof of Theorem 1.1 gives rise to Theorem 1.3. We also include a result

(Theorem 5.2) of Gilles Pisier, relating the constant $d_k(X)$ to the relative Euclidean factorization constant $e_k(X)$ of a Banach space X for all $k \geq 1$. This result is of independent interest as well.

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2. PRELIMINARIES

2.1. Group von Neumann algebras. Let Γ be a countable discrete group, and let $\lambda: \Gamma \rightarrow B(\ell^2(\Gamma))$ be its left regular representation, i.e., the representation of Γ given by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$ for $g, h \in \Gamma$ and $\xi \in B(\ell^2(\Gamma))$. The group von Neumann algebra $L(\Gamma)$ of Γ is given by the double commutant (in $B(\ell^2(\Gamma))$) of the set $\{\lambda(g) \mid g \in \Gamma\}$. The von Neumann algebra $L(\Gamma)$ has a normal faithful trace τ given by $\tau(x) = \langle x\delta_1, \delta_1 \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\ell^2(\Gamma)$. Group von Neumann algebras are important and motivating examples of von Neumann algebras.

2.2. Noncommutative L^p -spaces and their approximation properties. Let M be a finite von Neumann algebra with normal faithful trace τ . For $1 \leq p < \infty$, the noncommutative L^p -space $L^p(M, \tau)$ is the completion of M with respect to $\|x\|_p = \tau((x^*x)^{\frac{p}{2}})^{\frac{1}{p}}$. For $p = \infty$, we set $L^\infty(M, \tau) = M$. In this article, we deal with noncommutative L^p -spaces coming from group von Neumann algebras.

Noncommutative L^p -spaces are important examples of operator spaces. The operator space structure on a noncommutative L^p -space $L^p(M, \tau)$ can be obtained by realizing $L^p(M, \tau)$ as an interpolation space of the couple $(M, L^1(M, \tau))$ (see [15]). An operator space E has the completely bounded approximation property (CBAP) if there is a net $F_\alpha: E \rightarrow E$ of finite-rank maps with $\sup_\alpha \|F_\alpha\|_{cb} < \infty$ and $\lim_\alpha \|F_\alpha x - x\| = 0$ for all $x \in E$. An operator space E has operator space approximation property (OAP) if there exists a net F_α of finite-rank maps on E such that $\lim_\alpha \|(\text{id}_{\mathcal{K}(\ell^2)} \otimes F_\alpha)x - x\| = 0$ for all $x \in \mathcal{K}(\ell^2) \otimes_{\min} E$. If E has the CBAP, it also has the OAP.

2.3. The $\text{AP}_{p,cb}^{\text{Schur}}$. As mentioned in the introduction, we use the result of Lafforgue and the second named author that relates the CBAP of a noncommutative L^p -space to an approximation property of the underlying group. We recall this property below.

Recall that for a Hilbert space \mathcal{H} and $p \in [1, \infty)$, the Schatten class $S^p(\mathcal{H})$ is defined as the Banach space of bounded operators on \mathcal{H} such that $\|T\|_p := \text{Tr}(|T|^p)^{1/p} < \infty$, and $S^\infty(\mathcal{H})$ is the space $\mathcal{K}(\mathcal{H})$ consisting of compact operators. For a measure space (X, μ) , the class $S^2(L^2(X, \mu))$ can be identified with $L^2(X \times X, \mu \otimes \mu)$. Hence, a function $\psi \in L^\infty(X \times X, \mu \otimes \mu)$ induces a bounded map on $S^2(L^2(X, \mu))$ corresponding to multiplication on $L^2(X \times X, \mu \otimes \mu)$. The function ψ is said to be an S^p -multiplier if this map sends $S^p \cap S^2$ into S^p and extends to a bounded map on S^p . The norm of this map will be denoted by $\|\psi\|_{M(S^p)}$ and its completely bounded norm by $\|\psi\|_{cbM(S^p)}$.

In the situation that $(X, \mu) = (G, m)$ is a locally compact group with left Haar measure, a function $\varphi \in L^\infty(G, m)$ is said to be an S^p -multiplier if the function $(g, h) \mapsto \varphi(g^{-1}h)$ is an S^p -multiplier. The corresponding bounded linear map on $S^p(L^2(G, m))$ is called M_φ . Its norm is denoted by $\|\varphi\|_{M(S^p)}$ and its completely bounded norm by $\|\varphi\|_{cbM(S^p)}$.

Recall that the Fourier algebra $A(G)$ (see [10]) of a locally compact group G consists of the coefficients of the left regular representation of G : we have $\varphi \in A(G)$ if and only if there exist $\xi, \eta \in L^2(G)$ such that for all $x \in G$ we have $\varphi(x) = \langle \lambda(x)\xi, \eta \rangle$. The norm given by $\|\varphi\|_{A(G)} = \min\{\|\xi\|\|\eta\| \mid \forall x \in G \varphi(x) = \langle \lambda(x)\xi, \eta \rangle\}$, makes it into a Banach space.

Let $1 \leq p \leq \infty$. A locally compact group G is said to have the $\text{AP}_{p,cb}^{\text{Schur}}$ if there exists a net $(\varphi_\alpha)_\alpha$ in $A(G)$ such that $\sup_\alpha \|\varphi_\alpha\|_{cbM(S^p(L^2(G)))} < \infty$ and $\varphi_\alpha \rightarrow 1$ uniformly on compacta.

It is known (see [20, Theorem 2.5]) that if Γ is a lattice in a locally compact group G , then for $1 \leq p \leq \infty$, the group Γ has the $\text{AP}_{p,cb}^{\text{Schur}}$ if and only if G has the $\text{AP}_{p,cb}^{\text{Schur}}$. It is also known that the $\text{AP}_{p,cb}^{\text{Schur}}$ passes to closed subgroups (see [20, Proposition 2.3]) and that it is preserved under local isomorphisms of Lie groups with finite center (see [16, Proposition 3.11]).

We use the following result (see [20, Corollary 3.13]), relating the CBAP and the OAP to the $\text{AP}_{p,cb}^{\text{Schur}}$: if $p \in (1, \infty)$ and Γ is a countable discrete group such that $L^p(L(\Gamma))$ has the OAP, then Γ has the $\text{AP}_{p,cb}^{\text{Schur}}$ for that value of p . In particular, if $L^p(L(\Gamma))$ has the CBAP, then Γ has the $\text{AP}_{p,cb}^{\text{Schur}}$.

We summarize the above results in the following lemma, which is exactly what we use in this article.

Lemma 2.1. Let G be a locally compact group, and let Γ be a lattice in G . If $p \in (1, \infty)$ and G does not have the $\text{AP}_{p,cb}^{\text{Schur}}$ (for this p), then $L^p(L(\Gamma))$ does not have the CBAP or OAP.

For completeness, let us also mention a result from [13] (see also [14, Theorem 4.2]), asserting that the CBAP and OAP are equivalent for noncommutative L^p -spaces of a QWEP von Neumann algebra. This is not needed for what follows.

2.4. Expander families. The graphs we consider in this article are undirected, and we allow loops and multiple edges. Formally, this means that a graph is a pair $\mathcal{G} = (V, E)$ consisting of a set V of vertices and a multiset E of edges, i.e., an edge is a subset of V of cardinality 1 or 2. The degree of a vertex $v \in V$ is the number of edges that contain v (with this definition a loop counts as 1 for the degree). A graph is said to be k -regular for some $k \geq 1$ if the degree of every vertex is k . The vertex set of a graph can be considered as a metric space with respect to the graph distance.

All graphs we study are finite Schreier graphs of finitely generated groups. They are constructed from a group Γ with symmetric finite generating set S and a finite index subgroup $\Lambda < \Gamma$, and denoted by $(\Gamma/\Lambda, S)$. The set of vertices is the coset space Γ/Λ and an edge $\{v, w\}$ appears with multiplicity equal to the number of elements $s \in S$ such that $sv = w$ (this number is well-defined because S is symmetric). This defines a k -regular graph, where k is the number of elements of S . In the particular case when Λ is a normal subgroup, the Schreier graph $(\Gamma/\Lambda, S)$ is

actually a Cayley graph of the group Γ/Λ . There are no loops if $S \cap \Lambda = \emptyset$ and no multiple edges if S maps injectively into the quotient Γ/Λ .

Let $\mathcal{G} = (V, E)$ be a finite graph. The boundary ∂F of a set $F \subset V$ is defined by $\partial F = \{\{v, w\} \in E \mid v \in F, w \in V \setminus F\}$. The Cheeger constant $h(\mathcal{G})$ of the graph \mathcal{G} is defined by

$$h(\mathcal{G}) = \min \left\{ \frac{|\partial F|}{|F|} \mid F \subseteq V, 0 < |F| \leq \frac{1}{2}|V| \right\}.$$

A graph \mathcal{G} is connected if and only if $h(\mathcal{G}) > 0$.

An expander family is a sequence of finite graphs with strong connectivity properties, which are quantified by the Cheeger constant.

Definition 2.2. Let $k \geq 1$, and let $\{\mathcal{G}_n\}_{n \geq 1}$ be a sequence of finite k -regular graphs. Then $\{\mathcal{G}_n\}_{n \geq 1}$ is an expander family if $|V(\mathcal{G}_n)| \rightarrow \infty$ as $n \rightarrow \infty$ and if there is a $\varepsilon > 0$ such that $h(\mathcal{G}_n) \geq \varepsilon$ for all $n \geq 1$. Here, $V(\mathcal{G}_n)$ denotes the vertex set of \mathcal{G}_n .

For more details on expander families, we refer to [21]. Note that sometimes different conventions are used in the literature.

2.5. Non-coarse-embeddability. Let us first recall the definition of coarse embedding. A family of graphs X_i with induced distance d_i embeds coarsely into a metric space (Y, d) if there exist 1-Lipschitz maps $f_i: X_i \rightarrow Y$ and an increasing map $\rho: [0, \infty) \rightarrow [0, \infty)$ such that $\lim_{t \rightarrow \infty} \rho(t) = \infty$ and $\rho(d_i(x, y)) \leq d(f_i(x), f_i(y))$ for all i and all $x, y \in X_i$. We refer to [26] for an overview of the theory.

In this article, we essentially work on the level of the Lie group $\mathrm{SL}(n, \mathbb{R})$ rather than on the level of the Cayley graphs of $\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$. The non-coarse-embeddability of the associated expander family follows, by an argument of Lafforgue [18], from a Banach space version of property (T) for $\mathrm{SL}(n, \mathbb{R})$ (Theorem 5.8). Lafforgue's argument is an adaptation of the fact (due to Kazhdan and Margulis) that property (T) for $\mathrm{SL}(n, \mathbb{R})$ implies that the Cayley graphs of $\mathrm{SL}(n, \mathbb{Z}/i\mathbb{Z})$ form an expander family, and of Gromov's proof that such a family does not coarsely embed into a Hilbert space.

Let us recall Lafforgue's argument. Let G be a locally compact group, let Γ be lattice in G , let $(\Gamma_i)_i$ be a sequence of finite index subgroups in Γ with index tending to ∞ , and let X be a superreflexive Banach space. Let π_i denote the quasi-regular representation of Γ on $\ell^2(\Gamma/\Gamma_i; X)$ and consider the direct sum $\pi_X = \oplus_i \pi_i$, which is an isometric representation of Γ on $\ell^2(\sqcup_i \Gamma/\Gamma_i; X)$. Consider the induced representation $\mathrm{Ind}_\Gamma^G \pi_X$. Recall that the representation space of $\mathrm{Ind}_\Gamma^G \pi_X$ is the space X' of Bochner-measurable functions $f: G \rightarrow \ell^2(\sqcup_i \Gamma/\Gamma_i; X)$ satisfying $\|f\| = \left(\int_{G/\Gamma} \|f(g)\|^2 dg \right)^{1/2} < \infty$ and $f(g\gamma) = \pi_X(\gamma^{-1})f(g)$ for all $g \in G$ and $\gamma \in \Gamma$, and G acts by

$$(\mathrm{Ind}_\Gamma^G \pi_X)(h)f(g) = f(h^{-1}g).$$

This construction depends on the choice of Haar measure on G , and we choose the one that is normalized so that G/Γ has measure 1.

The space X'^G of invariant vectors for $\mathrm{Ind}_\Gamma^G \pi_X$ is the set of constant functions f from G to space of the invariant vectors for π_X , i.e., the space of $\xi \in \ell^2(\sqcup_i \Gamma/\Gamma_i; X)$ that are constant on each Γ/Γ_i . Also, X'^G has an invariant complement subspace Y' , namely the space of functions $f \in X'$ with values in the space $Y := \oplus_i \ell_0^2(\Gamma/\Gamma_i; X)$ consisting of $\xi \in \ell^2(\sqcup_i \Gamma/\Gamma_i; X)$ that have mean 0 on each Γ/Γ_i .

Let us assume that there exists a Borel measure ν on G such that $\nu(1) = 1$ and $\|\text{Ind}_\Gamma^G \pi_X(\nu)\|_{B(Y')} \leq \frac{1}{4}$. The existence of such a ν is typically provided by some Banach space version of property (T) (see Section 5). Let $\Omega \subset G$ be a Borel fundamental domain, i.e., for every $g \in G$, there is a unique $\gamma(g) \in \Gamma$ such that $g \in \Omega\gamma(g)$. The map from Y to Y' given by $f \mapsto \tilde{f}$, where $\tilde{f}(\omega\gamma) = \pi_X(\gamma)f$ for $\omega \in \Omega$ and $\gamma \in \Gamma$, is an isometry. Also, the map from Y' to Y given by $f \mapsto \int_\Omega f(\omega)d\omega$ has norm 1. Hence, the composition of these maps with $\text{Ind}_\Gamma^G \pi_X(\nu)$ has norm less than $\frac{1}{4}$ on Y and is equal to $\pi_X(\mu)$ for the probability measure μ on Γ satisfying $\mu(\gamma_0) = \int_\Omega \int_G 1_{\gamma(g^{-1}\omega)=\gamma_0} d\nu(g)d\omega$. By replacing μ by a measure with finite support and at total variation distance less than $\frac{1}{4}$ from μ , we may assume that $\|\pi_X(\mu)\|_{B(Y)} \leq \frac{1}{2}$. Hence, $\text{Id} - \pi_X(\mu)$ is invertible on Y and $\|(\text{Id} - \pi_X(\mu))^{-1}\| \leq 2$. This means that for every i and every $f_i: \Gamma/\Gamma_i \rightarrow X$ such that $\sum_{x \in \Gamma/\Gamma_i} f_i(x) = 0$, we have

$$(2) \quad \frac{1}{|\Gamma/\Gamma_i|} \sum_{x \in \Gamma/\Gamma_i} \|f_i(x)\|^2 \leq 4 \frac{1}{|\Gamma/\Gamma_i|} \sum_{x \in \Gamma/\Gamma_i} \left\| \int (f_i(x) - f_i(\gamma^{-1}x)) d\mu(\gamma) \right\|_X^2.$$

It is classical that this inequality implies that if S is a symmetric finite generating set in Γ , the family of graphs $(\Gamma/\Gamma_i, S)$ does not coarsely embed into X . Indeed, if the functions f_i are 1-Lipschitz, we have $\|f_i(x) - f_i(\gamma^{-1}x)\| \leq |\gamma|_S$ where $|\gamma|_S$ is the word-length of γ with respect to the generating set S , so that the right-hand side is bounded by $4K^2$, where $K = \max\{|\gamma|_S \mid \mu(\gamma) > 0\}$. If moreover f_i has mean zero (which can be achieved by subtracting from f_i its average), by (2) we get that $\|f_i(x)\| \leq 2\sqrt{2}K$ for at least half of the vertices in Γ/Γ_i . This prevents (f_i) to be a coarse embedding. Indeed, since the graph has bounded degree, the typical distance between two points in Γ/Γ_i is at least of order $\log(|\Gamma/\Gamma_i|)$, which tends to infinity.

For further use in the proof of Theorem 1.3, let us observe that the above representation $\text{Ind}_\Gamma^G \pi_X$ can be identified as the representation on $L^2(\sqcup_i (G \times_\Gamma \Gamma/\Gamma_i); X)$ coming from the measure-preserving action of G on $\sqcup_i (G \times_\Gamma \Gamma/\Gamma_i)$. Here $G \times_\Gamma \Gamma/\Gamma_i$ is the quotient of $G \times \Gamma/\Gamma_i$ by the equivalence relation $(g, x) \sim (g\gamma, \gamma^{-1}x)$ for all $g \in G$, $\gamma \in \Gamma$ and $x \in \Gamma/\Gamma_i$.

3. HARMONIC ANALYSIS ON THE $(n-1)$ -SPHERE

Fix $n \geq 3$. In what follows, constants, functions and operators may implicitly depend on n . Equip the sphere $\mathbb{S}^{n-1} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$ with the Lebesgue probability measure. For $\delta \in [-1, 1]$, let T_δ be the operator on $L^2(\mathbb{S}^{n-1})$ defined by $T_\delta f(x) =$ the average of f on $\{y \in \mathbb{S}^{n-1} \mid \langle x, y \rangle = \delta\}$. Equivalently, considering $\text{SO}(n-1)$ as the subgroup of $\text{SO}(n)$ fixing the first coordinate vector e_1 and using the identification $\mathbb{S}^{n-1} \cong \text{SO}(n-1) \backslash \text{SO}(n)$ through the map $\text{SO}(n-1)g \mapsto g^{-1}e_1$, we can consider $L^2(\mathbb{S}^{n-1})$ as a subspace of $L^2(\text{SO}(n))$. Then T_δ is the operator on $L^2(\text{SO}(n))$ equal to

$$(3) \quad \int_{\text{SO}(n-1) \times \text{SO}(n-1)} \lambda(ugu') dudu' \in B(L^2(\text{SO}(n)))$$

for $g \in \text{SO}(n)$ satisfying $g_{11} = \delta$. Here, λ denotes the left-regular representation.

Proposition 3.1. For $|\delta| < 1$, the operator T_δ belongs to $S^p(L^2(\mathbb{S}^{n-1}))$ if $p > 2 + \frac{2}{n-2}$. Moreover, for such p there exist constants $C_p \geq 2$ and $\alpha_p \in (0, 1)$ such

that for all $\delta \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\|T_0 - T_\delta\|_{S^p(L^2(\mathbb{S}^{n-1}))} \leq C_p |\delta|^{\alpha_p}.$$

The case $n = 3$ was proved in [20, Lemma 5.3]. For the proof of Proposition 3.1, we use some facts from the representation theory of $\mathrm{SO}(n)$ (see for example [7, Section 7.2–7.4]) that we collect in the following lemma.

Lemma 3.2. There is an orthogonal decomposition $L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^\infty \mathcal{H}_k$ such that each \mathcal{H}_k has finite dimension

$$m_k = \frac{(n+k-3)!(n+2k-2)}{(n-2)!k!}.$$

Moreover, the operators T_δ are diagonal with respect to this decomposition, and $T_\delta|_{\mathcal{H}_k} = \varphi_k(\delta) \mathrm{Id}_{\mathcal{H}_k}$, where φ_k is given by the formula

$$\varphi_k(x) = c_n \int_0^\pi (x + i\sqrt{1-x^2} \cos \varphi)^k \sin^{n-3} \varphi d\varphi$$

for $x \in [-1, 1]$. Here, $c_n = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{\pi} \Gamma(\frac{n-2}{2})}$, so that $\varphi_k(1) = 1$.

Remark 3.3. This lemma expresses the fact that $(\mathrm{SO}(n), \mathrm{SO}(n-1))$ is a Gelfand pair with spherical functions $g \mapsto \varphi_k(g_{11})$. For this Gelfand pair, these functions are Gegenbauer (also called ultraspherical) polynomials. The spaces \mathcal{H}_k are distinct irreducible representations of $\mathrm{SO}(n)$, and the operators T_δ commute with the representation of $\mathrm{SO}(n)$, so that Schur's Lemma implies that they are diagonal in the decomposition $\bigoplus_k \mathcal{H}_k$. The value $\varphi_k(\delta)$ can be computed by considering the harmonic polynomial $(x_1 + ix_2)^k \in \mathcal{H}_k$.

Lemma 3.4. There exists a constant C such that for all $k \geq 1$ and $x \in (-1, 1)$,

$$|\varphi_k(x)| \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}} \quad \text{and} \quad |\varphi'_k(x)| \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}} \frac{k}{\sqrt{1-x^2}}.$$

Proof. Since $\varphi_1(x) = x$, we can assume $k \geq 2$. We claim that there is a constant C (depending on n) such that for $k \geq 1$,

$$(4) \quad \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \leq \frac{C}{(k(1-x^2))^{\frac{n-2}{2}}}.$$

This implies the two inequalities of the lemma (for a different value of C). The first inequality is immediate (and holds already for $k \geq 1$), and for the second one, use

$$\begin{aligned} |\varphi'_k(x)| &= c_n \left| \int_0^\pi k \left(1 - i \frac{x \cos \varphi}{\sqrt{1-x^2}} \right) (x + i\sqrt{1-x^2} \cos \varphi)^{k-1} \sin^{n-3} \varphi d\varphi \right| \\ &\leq c_n \frac{k}{\sqrt{1-x^2}} \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^{k-1} \sin^{n-3} \varphi d\varphi. \end{aligned}$$

Let us prove (4). Firstly, note that

$$|x + i\sqrt{1-x^2} \cos \varphi|^2 = x^2 + (1-x^2) \cos^2 \varphi = 1 - (1-x^2) \sin^2 \varphi \leq e^{-(1-x^2) \sin^2 \varphi}.$$

This implies

$$\int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \leq 2 \int_0^{\frac{\pi}{2}} e^{-\frac{k}{2}(1-x^2) \sin^2 \varphi} \sin^{n-3} \varphi d\varphi.$$

Cut the integral into two pieces as $\int_0^{\frac{\pi}{2}} = \int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}$. For $\varphi \in [\frac{\pi}{4}, \frac{\pi}{2}]$, estimate $e^{-\frac{k}{2}(1-x^2) \sin^2 \varphi} \sin^{n-3} \varphi$ by $e^{-\frac{k}{4}(1-x^2)}$ to dominate the second integral by $\frac{\pi}{4} e^{-\frac{k}{4}(1-x^2)}$.

For the first integral, substitute $t = \sqrt{k(1-x^2)} \sin \varphi$ and use $d(\sin \varphi) = \cos \varphi d\varphi \geq \frac{1}{\sqrt{2}} d\varphi$ to dominate the first integral by

$$\frac{\sqrt{2}}{(k(1-x^2))^{\frac{n-2}{2}}} \int_0^{\sqrt{\frac{k}{2}(1-x^2)}} e^{-\frac{t^2}{2}} t^{n-3} dt.$$

These two inequalities together become

$$\begin{aligned} \int_0^\pi |x + i\sqrt{1-x^2} \cos \varphi|^k \sin^{n-3} \varphi d\varphi \\ \leq \frac{2\sqrt{2}}{(k(1-x^2))^{\frac{n-2}{2}}} \int_0^\infty e^{-\frac{t^2}{2}} t^{n-3} dt + \frac{\pi}{2} e^{-\frac{k}{4}(1-x^2)}, \end{aligned}$$

which implies (4). \square

Proof of Proposition 3.1. By Lemma 3.2, we have

$$\|T_x\|_{S^p}^p = \sum_{k \geq 0} m_k |\varphi_k(x)|^p \quad \text{and} \quad \|T_0 - T_x\|_{S^p}^p = \sum_{k \geq 1} m_k |\varphi_k(0) - \varphi_k(x)|^p.$$

By the formula for m_k , there exists an $A > 0$ (depending on n) such that $m_k \leq Ak^{n-2}$ for $k \geq 1$. Hence, by Lemma 3.4, we have $m_k |\varphi_k(x)|^p \leq C(x, n, p) k^{n-2-p\frac{n-2}{2}}$ for some constant $C(x, n, p)$ depending on x , n and p . We conclude that $T_x \in S^p$ if $p > 2 + \frac{2}{n-2}$ and $x \in (-1, 1)$, because the series $\sum_{k \geq 1} k^{n-2-p\frac{n-2}{2}}$ converges if $n - 2 - p\frac{n-2}{2} < -1$, i.e., $p > 2 + \frac{2}{n-2}$. For the second estimate, assume that $x \in [-\frac{1}{2}, \frac{1}{2}]$. Using Lemma 3.4, we dominate $|\varphi_k(0) - \varphi_k(x)|$ by $|x| \sup_{y \in [0, x]} |\varphi'_k(y)|$ for small values of k , and by $|\varphi_k(0)| + |\varphi_k(x)|$ for large values of k . More precisely, we obtain a constant $C > 0$ (depending on n) such that for $k \geq 1$ and $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$|\varphi_k(x) - \varphi_k(0)| \leq \frac{C}{k^{\frac{n-2}{2}}} \min\{1, k|x|\}.$$

The proposition now follows from a simple computation. \square

3.1. Consequences in terms of S^p -multipliers. In Section 4, Proposition 3.1 will be used in the following form.

Lemma 3.5. Let $\varphi: \text{SO}(n, \mathbb{R}) \rightarrow \mathbb{C}$ be a continuous $\text{SO}(n-1)$ -bi-invariant function that is a multiplier of $S^p(L^2(\text{SO}(n)))$. If $g, g' \in \text{SO}(n)$, then

$$|\varphi(g) - \varphi(g')| \leq 2C_p \max(|g_{11}|^{\alpha_p}, |g'_{11}|^{\alpha_p}) \|\varphi\|_{M(S^p)}.$$

Proof. Let $g, g' \in \text{SO}(n)$, and let $\delta = g_{11}$ and $\delta' = g'_{11}$. If $\max(|\delta|, |\delta'|) \geq \frac{1}{2}$, then $|\varphi(g) - \varphi(g')| \leq 2\|\varphi\|_{L^\infty} \leq 2\|\varphi\|_{M(S^p)}$, and the claim follows, since $2^{-\alpha_p} C_p \geq 1$. Therefore, the result follows from the following inequality: if $|\delta|, |\delta'| < 1$,

$$|\varphi(g) - \varphi(g')| \leq \|\varphi\|_{M(S^p)} \|T_\delta - T_{\delta'}\|_{S^p}.$$

We give two proofs of this inequality. Firstly, we consider again the operators T_δ on $\mathcal{H} = L^2(\text{SO}(n))$ given by (3). Then for every $g \in \text{SO}(n)$, we have $M_\varphi(T_{g_{11}}) = \varphi(g)T_{g_{11}}$. This equality is a particular case of a more general fact: for every $\text{SO}(n-1)$ -bi-invariant probability measure μ on $\text{SO}(n)$ with support contained in $\{g \mid |g_{1,1}| < 1\}$, we have $\lambda(\mu) \in S^p(L^2(\text{SO}(n)))$ and $M_\varphi(\lambda(\mu)) = \lambda(\varphi\mu)$. If μ is absolutely continuous with respect to the Haar measure with a density in $L^2(\text{SO}(n))$, then $\lambda(\mu) \in S^2(L^2(\text{SO}(n)))$ and the equality $M_\varphi(\lambda(\mu)) = \lambda(\varphi\mu)$ is the very definition of M_φ (this does not use that μ is $\text{SO}(n-1)$ -bi-invariant). The general case follows by a density argument, using that by Proposition 3.1, for any

$\varepsilon > 0$, the map $\mu \mapsto \lambda(\mu)$ is continuous from the set of $\mathrm{SO}(n-1)$ -bi-invariant probability measures with support in $\{g \mid |g_{1,1}| \leq 1 - \varepsilon\}$ equipped with the weak-* topology to $S^p(L^2(\mathrm{SO}(n)))$ equipped with the norm topology. The fact that the T_δ 's have a common eigenvector with eigenvalue 1, namely a constant function on $L^2(\mathrm{SO}(n))$, implies that

$$|\varphi(g) - \varphi(g')| \leq \|\varphi(g)T_\delta - \varphi(g')T_{\delta'}\|_{S^p} \leq \|\varphi\|_{M(S^p)}\|T_\delta - T_{\delta'}\|_{S^p},$$

which proves the claim.

A dual proof proceeds along the lines of [16, Section 3]. Let $q = \frac{p}{p-1}$ be the conjugate exponent of p . By [16, Proposition 2.7], we can write $\varphi(g) = \sum_k c_k m_k \varphi_k(g_{11})$, where the c_k 's play the role of Fourier coefficients and satisfy $(\sum_k m_k |c_k|^q)^{\frac{1}{q}} \leq \|\varphi\|_{M(S^q)} = \|\varphi\|_{M(S^p)}$. Hence, by the Hölder inequality,

$$|\varphi(g) - \varphi(g')| \leq \|\varphi\|_{M(S^p)} \left(\sum_k m_k |\varphi_k(\delta) - \varphi_k(\delta')|^p \right)^{\frac{1}{p}} = \|\varphi\|_{M(S^p)} \|T_\delta - T_{\delta'}\|_{S^p}.$$

□

4. APPROXIMATION PROPERTIES FOR $L^p(L(\mathrm{SL}(r, \mathbb{Z})))$.

In this section, we will prove Theorem 1.1. In fact, we will obtain a more general statement (Theorem 4.7), namely that certain noncommutative L^p -spaces associated with arbitrary lattices in connected high rank Lie groups do not have the CBAP. These results will follow from the following theorem.

Theorem 4.1. Let $n \geq 3$, let $r \geq 2n - 3$ and $p \in [1, 2 - \frac{2}{n}] \cup (2 + \frac{2}{n-2}, \infty]$. Then there does not exist a sequence of functions $\varphi_n \in C_0(\mathrm{SL}(r, \mathbb{R}))$ such that $\sup_n \|\varphi_n\|_{M(S^p)} < \infty$ and

$$\lim_n \varphi_n(g) = 1 \text{ for all } g \in \mathrm{SL}(r, \mathbb{R}).$$

Since for any locally compact group G , we have $A(G) \subset C_0(G)$, and since pointwise convergence is weaker than uniform convergence on compacta, the above result implies that $\mathrm{SL}(r, \mathbb{R})$ does not have the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$. As is mentioned in the introduction and more precisely in Lemma 2.1, a consequence of this is that for p and r as given in the theorem and a lattice Γ in $\mathrm{SL}(r, \mathbb{Z})$, the noncommutative L^p -space $L^p(L(\Gamma))$ does not have the CBAP, i.e., Theorem 1.1 follows directly.

The strategy of proving Theorem 4.1 is based on the approach of [20] and [11, Section 5]. Firstly, Proposition 3.1 gives rise to certain local Hölder continuity estimates for $\mathrm{SO}(n)$ -bi-invariant S^p -multipliers on $\mathrm{SL}(n, \mathbb{R})$, as given in Lemma 4.3. The next step is to find a path going to infinity in the Weyl chamber of $\mathrm{SL}(r, \mathbb{R})$ for r large enough by combining such local estimates. It turns out that $r = 2n - 3$ is enough. We now make this precise.

Fix $n \geq 3$. By embedding $\mathrm{SL}(2n - 3, \mathbb{R})$ into $\mathrm{SL}(r, \mathbb{R})$ for $r \geq 2n - 3$, we see that it is enough to consider the case $r = 2n - 3$. Also, by duality, we can assume that $p > 2 + \frac{2}{n-2}$. Then the Theorem 4.1 follows from an averaging argument and Proposition 4.2 below.

For $t, u, v \in \mathbb{R}$ with $t + \frac{u}{n-2} + v = 0$, we use the notation

$$(5) \quad D(v, u, t) = \mathrm{diag}(e^v, \dots, e^v, e^u, e^t, \dots, e^t) \in \mathrm{SL}(2n - 3, \mathbb{R})$$

for the diagonal matrix with $n - 2$ diagonal entries equal to e^v , 1 diagonal entry equal to e^u and $n - 2$ diagonal entries equal to e^t .

Proposition 4.2. For $p > 2 + \frac{2}{n-2}$, there is a function $\varepsilon_p \in C_0(\mathbb{R}_+)$ such that for every $\text{SO}(2n-3)$ -bi-invariant multiplier $\varphi: \text{SL}(2n-3, \mathbb{R}) \rightarrow \mathbb{C}$ of $S^p(L^2(\text{SL}(2n-3, \mathbb{R})))$, the function $\varphi(D(t, 0, -t))$ has a limit c for $t \rightarrow \infty$, and

$$|\varphi(D(t, 0, -t)) - c| \leq \varepsilon_p(t) \|\varphi\|_{M(S^p)}.$$

The crucial step to prove this proposition is the following lemma.

Lemma 4.3. Let $\varphi: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{C}$ be a multiplier of $S^p(L^2(\text{GL}(n, \mathbb{R})))$ that is $\text{SO}(n)$ -bi-invariant, and let $t < u < v \in \mathbb{R}$. Then for $0 < \delta < u - t$, we have

$$\begin{aligned} & |\varphi(\text{diag}(e^v, e^u, e^t, \dots, e^t)) - \varphi(\text{diag}(e^{v+\delta}, e^{u-\delta}, e^t, \dots, e^t))| \\ & \leq 2C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}. \end{aligned}$$

Proof. Let $u' = u - \delta$ and $v' = v + \delta$, and let $s = v + u - t$. Then $u, v, u', v' \in (t, s)$ and $u + v = u' + v' = s + t$. Consider the matrix $D = \text{diag}(e^{\frac{s}{2}}, e^{\frac{t}{2}}, \dots, e^{\frac{t}{2}}) \in \text{GL}(n, \mathbb{R})$. The map $g \in \text{SO}(n) \mapsto \varphi(DgD)$ is an $\text{SO}(n-1)$ -bi-invariant multiplier of $S^p(L^2(\text{SO}(n)))$ of norm less than $\|\varphi\|_{M(S^p)}$, so that by Lemma 3.5,

$$|\varphi(DgD) - \varphi(Dg'D)| \leq 2C_p \max(|g_{11}|^{\alpha_p}, |g'_{11}|^{\alpha_p}) \|\varphi\|_{M(S^p)}.$$

Let now g (resp. g') be a rotation of angle θ (resp. θ') in the plane generated by the two first coordinate vectors of \mathbb{R}^n , so that $g_{11} = \cos \theta$ and $g'_{11} = \cos \theta'$. Then $\varphi(DgD) = \varphi(\text{diag}(e^x, e^y, e^t, \dots, e^t))$ where $x \geq y$ are determined by

$$\begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix} \in \text{SO}(2) \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \text{SO}(2).$$

By a simple computation (see also [18, Lemme 2.8] or [11, Lemma 5.5]), there is a θ such that $x = v, y = u$ and $|\cos \theta| \leq e^{v-s} = e^{t-u}$. Similarly, there is θ' such that $|\cos \theta'| \leq e^{v+\delta-s} = e^{t+\delta-u}$ and such that $\varphi(Dg'D) = \varphi(\text{diag}(e^{v+\delta}, e^{u-\delta}, e^t, \dots, e^t))$. This proves the lemma. \square

Lemma 4.4. Let $\varphi: \text{SL}(2n-3, \mathbb{R}) \rightarrow \mathbb{C}$ be an $\text{SO}(2n-3)$ -bi-invariant multiplier of $S^p(L^2(\text{SL}(2n-3, \mathbb{R})))$, and let $t < u < v \in \mathbb{R}$ with $t + \frac{u}{n-2} + v = 0$. Then for $0 < \delta < u - t$, we have

$$|\varphi(D(v, u, t)) - \varphi(D(v + \frac{\delta}{n-2}, u - \delta, t))| \leq 2(n-2)C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}.$$

Proof. By the triangle inequality, we can write

$$|\varphi(D(v, u, t)) - \varphi(D(v + \frac{\delta}{n-2}, u - \delta, t))| \leq \sum_{k=1}^{n-2} |\varphi(D_{k-1}) - \varphi(D_k)|,$$

where D_k is the diagonal matrix with $(n-2-k)$ eigenvalues equal to e^v , k eigenvalues equal to $e^{v+\frac{\delta}{n-2}}$, one eigenvalue equal to $e^{u-\frac{k}{n-2}\delta}$, and $n-2$ eigenvalues equal to e^t . By Lemma 4.3, for each k the term $|\varphi(D_{k-1}) - \varphi(D_k)|$ is less than $2C_p e^{-\alpha_p(u-t-\frac{k}{n-2}\delta)} \|\varphi\|_{M(S^p)}$, which is less than $2C_p e^{-\alpha_p(u-t-\delta)} \|\varphi\|_{M(S^p)}$ and proves the lemma. \square

By conjugating by the Cartan automorphism $g \mapsto (g^t)^{-1}$, we get the following.

Lemma 4.5. Let $\varphi: \mathrm{SL}(2n-3, \mathbb{R}) \rightarrow \mathbb{C}$ be an $\mathrm{SO}(2n-3)$ -bi-invariant multiplier of $S^p(L^2(\mathrm{SL}(2n-3, \mathbb{R})))$, and let $t < u < v \in \mathbb{R}$ with $t + \frac{u}{n-2} + v = 0$. Then for $0 < \delta < v - u$, we have

$$|\varphi(D(v, u, t)) - \varphi(D(v, u + \delta, t - \frac{\delta}{n-2}))| \leq 2(n-2)C_p e^{-\alpha_p(v-u-\delta)} \|\varphi\|_{M(S^p)}.$$

Proof of Proposition 4.2. By combining the above two lemmas, we get that for $0 < \delta < v$,

$$|\varphi(D(v, 0, -v)) - \varphi(D(v + \frac{\delta}{n-2}, 0, -v - \frac{\delta}{n-2}))| \leq 4(n-2)C_p e^{-\alpha_p(v-\delta)} \|\varphi\|_{M(S^p)}.$$

This implies that $\varphi(D(t, 0, -t))$ satisfies the Cauchy criterion, and hence has a limit. Indeed, for $t \leq s$, define $\delta = \frac{t}{2}$ and the sequence (v_k) by $v_0 = t$ and $v_{k+1} = \min(s, v_k + \frac{\delta}{n-2})$ (so that $v_k = \min\{s, (1 + \frac{k}{2n-4})t\}$). If N is the first index such that $v_N = s$, then

$$\begin{aligned} |\varphi(D(t, 0, -t)) - \varphi(D(s, 0, -s))| &\leq \sum_{k=0}^{N-1} |\varphi(D(v_k, 0, -v_k)) - \varphi(D(v_{k+1}, 0, -v_{k+1}))| \\ &\leq \sum_{k=0}^{N-1} 4(n-2)C_p e^{-\alpha_p(\frac{1}{2} + \frac{k}{2n-4})t} \leq 4(n-2)C_p \frac{e^{-\alpha_p \frac{t}{2}}}{1 - e^{-\alpha_p \frac{t}{2n-4}}}. \end{aligned}$$

This proves Proposition 4.2. \square

We can now generalize our result to higher rank simple Lie groups. We will assume that the real rank is at least 9, so that we only need to consider the classical Lie groups, since all exceptional Lie groups have real rank 8 or less.

Lemma 4.6. Let G be a connected simple real Lie group with real rank $N \geq 9$. Then G contains a connected closed subgroup H locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$.

Proof. Since we assume the real rank of G to be at least 9, the group G is a classical Lie group. By Dynkin's classification of regular semisimple Lie subalgebras of semisimple Lie algebras, it is known that every simple Lie algebra of rank $N \geq 9$ contains \mathfrak{sl}_N as a Lie subalgebra [8] (see [9] for a translation). This Lie subalgebra gives rise to a connected Lie subgroup H in G that is locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$, and by a result of Mostow, it is closed, since G has discrete center [25, last theorem on p. 614] (see also [6, Corollary 1]). \square

Theorem 4.7. Let $n \geq 6$, let $N \geq 2n-3$, and let Γ be a lattice in a connected simple real Lie group with real rank at least N . Then the noncommutative L^p -space $L^p(L(\Gamma))$ does not have the CBAP for $p \in [1, 2 - \frac{2}{n}) \cup (2 + \frac{2}{n-2}, \infty]$.

Proof. Let $H \subset G$ be a closed subgroup locally isomorphic to $\mathrm{SL}(N, \mathbb{R})$ given by Lemma 4.6. Then H has finite center, because the fundamental group of $\mathrm{SL}(N, \mathbb{R})$ is finite. Hence, H does not have the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$, since $\mathrm{SL}(N, \mathbb{R})$ does not have it (see Section 2.3). Since the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ passes to closed subgroups (see Section 2.3), the group G does not have the $\mathrm{AP}_{p, \mathrm{cb}}^{\mathrm{Schur}}$ either. The result now follows by applying Lemma 2.1. \square

Remark 4.8. Actually, a stronger statement is true, namely, for Γ and p as above, the space $L^p(L(\Gamma))$ does not have the operator space approximation property (OAP) (see section 2 for the definition). By Lemma 2.1, this follows in the same way from Theorem 4.1 as Theorems 1.1 and 4.7.

5. NON-COARSE-EMBEDDABILITY OF EXPANDER FAMILIES

5.1. Conventions. In what follows, we only consider real Banach spaces.

Let G be a locally compact group, let X be a Banach space, and let $O(X)$ denote the group of all invertible linear isometries from X to X . In what follows, by an isometric representation of G on X , we mean a homomorphism $\pi: G \rightarrow O(X)$ that is strongly continuous, i.e., the map $G \rightarrow X$ given by $g \mapsto \pi(g)x$ is continuous for every $x \in X$. It is known that strong continuity for such representations is equivalent to other forms of continuity [24, Section 3.3] (see also [1, Lemma 2.4]).

If $\pi: G \rightarrow B(X)$ is a strongly continuous representation of a locally compact group G on a Banach space X and m is a compactly supported signed Borel measure on G , then by $\pi(m)$ we denote the operator in $B(X)$ given by

$$\pi(m)\xi = \int_G \pi(g)\xi dm(g) \text{ (Bochner integral) for } \xi \in X.$$

5.2. Versions of property (T) relative to Banach spaces. Let X be a Banach space. As defined in [1], a locally compact group G has property (T_X) if for every isometric representation $\pi: G \rightarrow O(X)$, the quotient representation $\pi': G \rightarrow O(X/X^{\pi(G)})$ does not have almost invariant vectors. Here, $X^{\pi(G)}$ denotes the closed subspace of X consisting of vectors that are fixed by π .

Let \mathcal{E} be a class of Banach spaces, and let G be a locally compact group. Consider the completion $C_{\mathcal{E}}(G)$ of $C_c(G)$ with respect to the norm $\sup \|\pi(f)\|_{B(X)}$, where the supremum is taken over all isometric representations of G on a space X in \mathcal{E} . Following [18, Section 4] and [19, Définition 0.4], we say that a group G has property $(T_{\mathcal{E}}^{\text{proj}})$ if $C_{\mathcal{E}}(G)$ contains an idempotent p such that for all isometric representations π of G on a space X in \mathcal{E} , the operator $\pi(p)$ is a projection on $X^{\pi(G)}$. Concretely, this is equivalent to the existence of a sequence $(m_n)_n$ of compactly supported signed measures on G such that $m_n(G) = 1$ for all $n \in \mathbb{N}$ and the sequence $(\pi(m_n))_n$ converges in the norm topology of $B(X)$ and uniformly in (π, X) to a projection on $X^{\pi(G)}$. We refer to [30, Section 2.9] for the equivalence. Property $(T_{\mathcal{E}}^{\text{proj}})$ is what we use in order to prove nonembeddability results for expander families.

It is clear that property $(T_{\mathcal{E}}^{\text{proj}})$ implies property (T_X) for all $X \in \mathcal{E}$. The converse does not hold in general. Indeed, every group G with Kazhdan's property (T) has property $(T_{L^1(G)})$ [2], but no infinite group has property $(T_{\mathcal{E}}^{\text{proj}})$ if $L^1(G) \in \mathcal{E}$. However, if X is superreflexive, property (T_X) is equivalent with a “non-uniform” version of property $(T_{\mathcal{E}}^{\text{proj}})$, as is shown by the following result.

Proposition 5.1. Let X be a superreflexive Banach space. If a discrete group G has property (T_X) , then for every isometric representation $\pi: G \rightarrow O(X)$, there is a projection onto the space $X^{\pi(G)}$ in the norm closure of

$$\{\pi(m) \mid m \text{ is a compactly supported probability measure on } G\}.$$

If a locally compact group G has property $(T_{\ell^2(X)})$, then for every isometric representation of G , such a projection also exists .

Proof. Let $\pi: G \rightarrow O(X)$ be an isometric representation. By [1, Proposition 2.3], we can assume that the norm on X is uniformly convex and uniformly smooth. By [1, Proposition 2.6], it follows that $X^{\pi(G)}$ has a G -invariant closed complemented subspace Y . We will construct a compactly supported probability measure m on G such that $\|\text{Id} + \pi(m)\|_{B(Y)} < 2$, where Id denotes the identity operator on Y .

Replacing m by the measure $A \mapsto \frac{1}{2}(m(A) + m(A^{-1}))$, we can assume that m is symmetric. For $n \geq 1$, define $m_n = (\frac{1}{2}\delta_1 + \frac{1}{2}m)^{*n}$, which is a compactly supported symmetric probability measure, and the sequence $\pi(m_n)$ converges to a projection onto $X^{\pi(G)}$ as $n \rightarrow \infty$.

If G is discrete, by property (T_X) we know that Y does not almost have invariant vectors, i.e., there is a finite subset $S \subset G \setminus \{1\}$ and $\varepsilon > 0$ such that $\sup_{s \in S} \|s \cdot \xi - \xi\| \geq \varepsilon$ for all unit vectors $\xi \in Y$. By the uniform convexity of X , this implies that there is a $\delta > 0$ such that $\inf_{s \in S} \|s \cdot \xi + \xi\| < 2 - \delta$. This implies that $\sum_{s \in S} (\text{Id} + \pi(s))$ has norm less than $2|S| - \delta$ on Y . In other words, if m is the uniform probability measure on S , then $\|\text{Id} + \pi(m)\|_{B(Y)} < 2$.

If G is locally compact, similarly by property $(T_{\ell^2(X)})$ there is $\delta > 0$ and a compact subset $Q \subset G$ such that $\inf_{s \in Q} \|s \cdot \xi + \xi\| < 2 - \delta$ for all unit vector $\xi \in \ell^2(Y)$. Consider the convex hull $C \subset C(Q)$ of the functions of the form $s \mapsto \langle s \cdot \xi + \xi, \eta \rangle$ for ξ and η in the unit balls of Y and Y^* , respectively. If $g = \sum \lambda_i \langle s \cdot \xi_i + \xi_i, \eta_i \rangle \in C$, we can write $g = \langle s \cdot \underline{\xi} + \underline{\xi}, \underline{\eta} \rangle$ for the unit vectors $\underline{\xi} = (\sqrt{\lambda_i} \xi_i)_i \in \ell^2(Y)$ and $\underline{\eta} = (\sqrt{\lambda_i} \eta_i)_i \in \ell^2(Y^*)$ and deduce that $\inf_Q g < 2 - \delta$. Hence, by the Hahn-Banach Theorem, there is a probability measure m on Q such that $\int g dm \leq 2 - \delta$ for all $g \in C$. This implies that $\|\text{Id} + \pi(m)\|_{B(Y)} \leq 2 - \delta$. \square

5.3. On the geometry of Banach spaces. We now give some background on condition (1). We refer to [32] for more information. Then we derive some consequences of (1) and Proposition 3.1.

Two Banach spaces X, Y are said to be C -isomorphic if there is an isomorphism $u: X \rightarrow Y$ such that $\|u\| \|u^{-1}\| \leq C$. The infimum of such C is called the Banach-Mazur distance between X and Y , also called the isomorphism constant from X to Y , and is denoted by $d(X, Y)$. It is known that if X has dimension k , then $d(X, \ell_k^2) \leq k^{\frac{1}{2}}$ (indeed, this follows by considering *John's ellipsoid*). We have equality for $X = \ell_k^1$, i.e., $d(\ell_k^1, \ell_k^2) = k^{\frac{1}{2}}$ for all $k \geq 1$.

For a Banach space X , put $d_k(X) = \sup\{d(E, \ell_{\dim E}^2) \mid E \subset X, \dim E \leq k\}$. From the reminders above, the inequality $d_k(X) \leq k^{\frac{1}{2}}$ holds for every Banach space X and every k , and if ℓ^1 is finitely representable in X , i.e., X contains subspaces $(1 + \varepsilon)$ -isomorphic to ℓ_k^1 for every $\varepsilon > 0$ and every k , then $d_k(X) = k^{\frac{1}{2}}$. Milman and Wolfson [23] proved the converse: if $\limsup_k d_k(X) k^{-\frac{1}{2}} > 0$, then ℓ^1 is finitely representable in X . A consequence is that if $d_k(X) < k^{\frac{1}{2}}$ for some k , then $\lim_k d_k(X) k^{-\frac{1}{2}} = 0$.

The Banach spaces in which ℓ^1 is not finitely representable were historically called B -convex spaces and have been extensively studied. A superreflexive space, i.e., a Banach space X such that every Banach space finitely representable in X is reflexive, is clearly B -convex. In fact, a Banach space X is B -convex if and only if it has type > 1 if and only if it is K -convex if and only if $\lim_k d_k(X) k^{-\frac{1}{2}} = 0$. The rate of convergence to 0 of this quantity is not yet completely understood. It is known that it converges to 0 at least as fast as a power of $\log k$, and it is a well-known open problem whether it converges as a power of k , as in (1) (see [32, Problem 27.6]). Also, it is known that (1) holds if X has type p and cotype q satisfying $\frac{1}{p} - \frac{1}{q} < \frac{1}{2}$, which is particularly the case if X has type 2.

Following [28], for a Banach space X and a $k \in \mathbb{N}$, one sets

$$e_k(X) = \sup \|u \otimes \text{Id}_X\|_{B(\ell^2(X))},$$

where the supremum is taken over all linear maps $u: \ell^2 \rightarrow \ell^2$ of norm 1 and rank k . We will need a result by Pietsch, asserting that for a Banach space X and $\beta \leq \frac{1}{2}$, we have $\sup_k d_k(X)k^{-\beta} < \infty$ if and only if $\sup_k e_k(X)k^{-\beta} < \infty$ [27]. In fact, a closer relationship exists between the numbers $e_k(X)$ and $d_k(X)$:

$$(6) \quad d_k(X) \leq e_k(X) \leq 2d_k(X).$$

The first inequality is classical (see for example [32, §27]), whereas the second inequality has been long known to Pisier as a consequence of [31]. With his kind permission, we include a proof here.

Theorem 5.2 (Pisier). Let X be a real Banach space. For every $k \in \mathbb{N}$,

$$e_k(X) \leq 2 \sup_{a: \ell_k^2 \rightarrow \ell_k^2, \|a\| \leq 1} \|a \otimes Id_X\|_{\ell_k^2(X) \rightarrow \ell_k^2(X)}.$$

In particular, $e_k(X) \leq 2d_k(X)$ for all $k \in \mathbb{N}$.

Proof. We can assume that X is finite-dimensional. Let $C_k(X)$ denote the set of operators $u: X \rightarrow \ell_k^2$ of the form

$$u(x) = \sum_{i=1}^{\infty} \xi_i(x) b e_i,$$

where the $\xi_i \in X^*$ satisfy $\sum_{i=1}^{\infty} \|\xi_i\|^2 < 1$, the vectors e_i denote the standard orthonormal basis vectors of ℓ^2 , and $\|b: \ell^2 \rightarrow \ell_k^2\| < 1$ (actually it follows from the self-duality of the 2-summing norm [32, Proposition 9.10] that $C_k(X)$ is the set of operators $u: X \rightarrow \ell_k^2$ of 2-summing norm less than 1). Since every operator $a: \ell^2 \rightarrow \ell^2$ of norm 1 and rank k can be written as $a = b^*c$ for operators $b, c: \ell^2 \rightarrow \ell_k^2$ of norm 1, we have

$$\begin{aligned} e_k(X) &= \sup_{\|\xi\|_{\ell^2(X)} < 1, \|\eta\|_{\ell^2(X^*)} < 1, \|b, c: \ell^2 \rightarrow \ell_k^2\| < 1} \langle (c \otimes Id_X)(\eta), (b \otimes Id_{X^*})(\xi) \rangle \\ &= \sup_{u \in C_k(X), v \in C_k(X^*)} tr(uv^*). \end{aligned}$$

Similarly, let $D_k(X)$ denote the set of operators $u: X \rightarrow \ell_k^2$ of the form

$$u(x) = \sum_{i=1}^k \xi_i(x) b e_i,$$

where the $\xi_i \in X^*$ satisfy $\sum_{i=1}^k \|\xi_i\|^2 < 1$, the vectors e_i denote the standard orthonormal basis vectors of ℓ_k^2 , and $\|b: \ell_k^2 \rightarrow \ell_k^2\| < 1$. It follows that

$$\sup_{a: \ell_k^2 \rightarrow \ell_k^2, \|a\| \leq 1} \|a \otimes Id_X\|_{\ell_k^2(X) \rightarrow \ell_k^2(X)} = \sup_{u \in D_k(X), v \in D_k(X^*)} tr(uv^*).$$

We claim that $C_k(X) \subset \sqrt{2} \text{conv}(D_k(X))$. The claim clearly implies the theorem, and it is proved by duality. The polar of $C_k(X)$ coincides, with respect to the duality $\langle u, v \rangle = tr(uv)$, with the operators $v: \ell_k^2 \rightarrow X$ such that

$$\pi_2(v) = \sup_{\|b: \ell^2 \rightarrow \ell_k^2\| \leq 1} \left(\sum_i \|u b e_i\|^2 \right)^{\frac{1}{2}} \leq 1.$$

Similarly, the polar of $D_k(X)$ is the set of operators $v: \ell_k^2 \rightarrow X$ such that

$$\pi_2^{(k)}(v) = \sup_{\|b: \ell_k^2 \rightarrow \ell_k^2\| \leq 1} \left(\sum_{i=1}^k \|u b e_i\|^2 \right)^{\frac{1}{2}} \leq 1.$$

The claim therefore follows from [32, Theorem 18.4], in which the inequality $\pi_2(v) \leq \sqrt{2}\pi_2^{(k)}(v)$ is proved for every rank k linear map. \square

The following result is essentially [30, Proposition 3.3]. In fact, it is what its proof actually shows.

Proposition 5.3. Let X be a Banach space such that $\sup_k e_k(X)k^{-\beta} \leq C' < \infty$. Then for every $p < \beta^{-1}$, there is a constant $C_p(X)$ (depending on C' , p and β) such that

$$\|T \otimes \text{Id}_X\|_{B(L^2(\Omega; X))} \leq C_p(X) \|T\|_{S^p(L^2(\Omega))}$$

for every measure space (Ω, μ) and every operator $T: L^2(\Omega) \rightarrow L^2(\Omega)$ belonging to the Schatten class S^p .

As a consequence, we obtain the following result.

Lemma 5.4. Let $n \geq 3$, and let X be a Banach space for which there exist $C' > 0$ and $\beta < \frac{1}{2}(1 - \frac{1}{n-1})$ such that $d_k(X) \leq C'k^\beta$ for all k . Then there exist $C_X \in \mathbb{R}$ and $\alpha_X > 0$ such that for all $\delta, \delta' \in [-1, 1]$,

$$(7) \quad \|(T_\delta - T_{\delta'}) \otimes \text{Id}_X\|_{B(L^2(\text{SO}(n); X))} \leq C_X \max(|\delta|^{\alpha_X}, |\delta'|^{\alpha_X}).$$

Moreover, C_X and α_X depend on C' and β only.

Proof. By the triangle inequality, we can assume that $\delta' = 0$, and we can assume that $\delta \in [-\frac{1}{2}, \frac{1}{2}]$ (for $|\delta| \geq \frac{1}{2}$, use that $\|T_\delta \otimes \text{Id}_X\|_{B(L^2(\text{SO}(n); X))} \leq 1$).

Our assumption on X means (recall (6)) that there is some $\varepsilon > 0$ such that $\sup_k e_k(X)k^{\varepsilon - \frac{1}{q}} < \infty$, where $q = 2 + \frac{2}{n-2}$. Pick $p > q$ such that $\varepsilon - \frac{1}{q} > -\frac{1}{p}$. Then (7) follows from Proposition 3.1 and Proposition 5.3. \square

Remark 5.5. Since (7) behaves well with respect to complex interpolation (see for example [30, Lemma 3.1]), Lemma 5.4 also holds if X is isomorphic to a complex-interpolation space $[X_0, X_1]_\theta$ for some $0 < \theta < 1$, where X_0 is a space as in the lemma and X_1 is an arbitrary Banach space. It might be that the spaces obtained in this way for a fixed n are all the spaces satisfying (1).

Now we combine Lemma 5.4 with Fell's absorption principle for isometric representations of compact groups on Banach spaces (see [30, Proposition 2.7]), asserting that if $\pi: K \rightarrow O(X)$ is an isometric representation of a compact group K on a Banach space X , then for every signed Borel measure m on K , we have $\|\pi(m)\|_{B(X)} \leq \|\lambda(m)\|_{B(L^2(K; X))}$. Using the explicit expression for T_δ , this implies the following result.

Lemma 5.6. Let $n \geq 3$, and let X be a Banach space satisfying (7) for some C_X , $\alpha_X > 0$ and all $\delta, \delta' \in [-1, 1]$. Then for every linear isometric representation π of $\text{SO}(n)$ on X and all $\text{SO}(n-1)$ -invariant unit vectors $\xi \in X$ and $\eta \in X^*$, the function $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$ satisfies

$$|\varphi(g) - \varphi(g')| \leq C_X \max(|g_{1,1}|^{\alpha_X}, |g'_{1,1}|^{\alpha_X}) \|\xi\|_X \|\eta\|_{X^*}$$

for all $g, g' \in \text{SO}(n)$.

5.4. Explicit behaviour of coefficients. Proposition 4.2 followed from Lemma 3.5 with a proof of combinatorial nature. The only property of S^p -multipliers that was used is the following: if $\varphi: G \rightarrow \mathbb{C}$ is an S^p -multiplier on a locally compact group G and H is a closed subgroup of G and $g, g' \in G$, then the function on H given by $h \mapsto \varphi(hg'h')$ is an S^p -multiplier of the group H with norm not greater than $\|\varphi\|_{M(S^p)}$. The same property holds if, for a given Banach space X , the word *S^p -multiplier* is replaced by *coefficient of a continuous isometric representation on X* and $\|\varphi\|_{M(S^p)}$ is replaced by the norm equal to the infimum of $\|\xi\|_X \|\eta\|_{X^*}$ over all strongly continuous isometric representations π of G on X and all vectors ξ and η such that $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$. Therefore, we can deduce the following proposition from Lemma 5.6 with exactly the same proof as for Proposition 4.2.

Proposition 5.7. Let $n \geq 3$, and let X be a Banach space for which there exist $C_X \in \mathbb{R}$ and $\alpha_X > 0$ such that (7) holds for all $\delta, \delta' \in [-1, 1]$. Then there is a function $\varepsilon_X \in C_0(\mathbb{R}_+)$ (depending on C_X and α_X only) such that the following holds: for every linear isometric strongly continuous representation π of $\mathrm{SL}(2n-3, \mathbb{R})$ on X and all $\mathrm{SO}(2n-3)$ -invariant vectors $\xi \in X$ and $\eta \in X^*$, the function $\varphi(g) = \langle \pi(g)\xi, \eta \rangle$ satisfies the fact that $\varphi(D(t, 0, -t))$ has a limit c as $t \rightarrow \infty$, and

$$|\varphi(D(t, 0, -t)) - c| \leq \varepsilon_X(t) \|\xi\|_X \|\eta\|_{X^*}.$$

5.5. Non-coarse-embeddability. Proposition 5.7 leads to the following theorem.

Theorem 5.8. Let X be a superreflexive Banach space satisfying (1). Then there exists an integer $N \geq 9$ such that for every connected simple Lie group G of real rank at least N the following holds: there is a sequence of compactly supported symmetric probability measures m_l on G such that for every isometric representation $\pi: G \rightarrow O(X)$, the sequence $(\pi(m_l))$ converges to a projection on $X^{\pi(G)}$. Moreover, the integer N and the measures m_l depend only on the value of β in (1).

Remark 5.9. The above theorem directly implies that for every $\beta < \frac{1}{2}$, there exists an $N \geq 9$ such that every connected simple Lie group with real rank at least N has property $(T_{\mathcal{E}}^{\mathrm{proj}})$ with respect to the class of superreflexive Banach spaces satisfying (1) for the given value of β and a fixed $C > 0$.

Proof. By our assumption, there exist $C > 0$ and $\beta < \frac{1}{2}$ such that $d_k(X) \leq Ck^\beta$ for all k . Take n so that $\beta < \frac{1}{2}(1 - \frac{1}{n-1})$. Our main task is to prove the theorem for $\mathrm{SL}(2n-3, \mathbb{R})$. Let now $G = \mathrm{SL}(2n-3, \mathbb{R})$ and define a sequence of symmetric probability measures m_l by

$$m_l(f) = \iint_{\mathrm{SO}(2n-3) \times \mathrm{SO}(2n-3)} f(kD(l, 0, -l)k') dk dk',$$

where the notation $D(l, 0, -l)$ was introduced in (5). Let π be an isometric strongly continuous representation of G on X . By Proposition 5.7, $\pi(m_l)$ has a limit P in the norm topology of $B(X)$. We claim that P is a projection on $X^{\pi(G)}$. This is where we use the assumption that X is superreflexive. By [1, Proposition 2.3], we can assume that the norm on X is uniformly convex and uniformly smooth. It is clear that P acts as the identity on $X^{\pi(G)}$. By [1, Proposition 2.6], $X^{\pi(G)}$ has a G -invariant complement closed subspace. By replacing X by this complement subspace, we can assume that $X^{\pi(G)} = 0$, and we have to prove that $P = 0$. This

follows from the version of the Howe-Moore property proved by Shalom (see [1, Theorem 9.1]).

Now standard arguments (see, e.g., [3, Section 1.6] or [18, Section 4]) imply that the conclusion of the theorem holds for every connected simple real Lie group containing a closed subgroup locally isomorphic to $\mathrm{SL}(2n-3, \mathbb{R})$, and hence, by Lemma 4.6, for every simple real Lie group of real rank $\geq \max\{9, 2n-3\}$. \square

Remark 5.10. If X satisfies (1) but is not superreflexive, the conclusion of the above theorem still holds for the representations of the form $\pi \otimes 1_X$ on $L^2(\Omega, \mu; X)$, where π is a unitary representation on $L^2(\Omega, \mu)$ that comes from a measure-preserving action of G on a σ -finite measure space (Ω, μ) .

Indeed, as in the proof above, it is sufficient to show that if n is such that $\beta < \frac{1}{2}(1 - \frac{1}{n-1})$, any representation of $\mathrm{SL}(2n-3, \mathbb{R})$ of the form $\pi \otimes 1_X$ as above satisfies that $(\pi \otimes 1_X)(m_l) = \pi(m_l) \otimes 1_X$ converges in the norm topology to a projection onto the $\mathrm{SL}(2n-3, \mathbb{R})$ -invariant vectors. On the one hand, by Fubini's theorem, $e_k(L^2(\Omega; X)) = e_k(X)$ for all k , and hence, by (6), $L^2(\Omega, \mu; X)$ satisfies (1) for the same β as X . Proposition 5.7 then implies that $(\pi \otimes 1_X)(m_l)$ has a limit in the norm topology. On the other hand, by the Howe-Moore property for (unitary representations of) $\mathrm{SL}(2n-3, \mathbb{R})$, the sequence $\pi(m_l)$ converges in the weak operator topology to the orthogonal projection P on the G -invariant vectors on $L^2(\Omega, \mu)$, which shows that the limit of $\pi(m_l) \otimes 1_X$ is $P \otimes 1$, a projection onto the $\mathrm{SL}(2n-3, \mathbb{R})$ -invariant vectors.

By the argument recalled in Section 2.5, Theorem 5.8 (if X is superreflexive) or Remark 5.10 (if X not superreflexive) imply the following result.

Theorem 5.11. Let X be a Banach space satisfying (1). Then there exists a natural number $N \geq 9$ such that if Γ is a lattice in a connected simple real Lie group of real rank at least N , if $(\Gamma_i)_{i \in \mathbb{N}}$ is a sequence of finite-index subgroups of Γ such that $|\Gamma/\Gamma_i| \rightarrow \infty$ for $i \rightarrow \infty$ and if S is a symmetric finite generating set of Γ , then the sequence of Schreier graphs $(\Gamma/\Gamma_i, S)$ does not coarsely embed into X .

Theorem 1.3 follows as a particular case of Theorem 5.11.

Remark 5.12. (Remark added after publication.) By using Veech's version of the Howe-Moore property [33], which holds in the more general setting of reflexive Banach spaces rather than superreflexive ones, the superreflexivity assumption in Theorems 1.4 and 5.8 can be replaced by the assumption that X is reflexive.

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